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Energy conservation, time-reversal invariance and reciprocity in ducts with flow

By WILLI MÖHRING

Max-Planck-Institut für Strömungsforschung, Bunsenstr. 10, 37075 Göttingen, Germany
e-mail: wmoehri@gwdg.de

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A sound wave propagating in an inhomogeneous duct consisting of two semi-infinite uniform ducts with a smooth transition region in between and which carries a steady flow is considered. The duct walls may be rigid or compliant. For an irrotational sound wave it is shown that the three properties of the title are closely related, such that the validity of any two implies the validity of the third. Furthermore it is shown that the three properties are fulfilled for lossless locally reacting duct walls provided the impedance varies at most continuously. For piecewise-continuous wall properties edge conditions are essential. By an analytic continuation argument it is shown that reciprocity remains true for walls with loss. For rotational flow, energy conservation theorems have been derived only with the help of additional potential-like variables. The inter-relation between the three properties remains valid if one considers these additional variables to be known. If only the basic gasdynamic variables in both half-ducts are known, one cannot formulate an energy conservation equation; however, reciprocity is fulfilled.

1. Introduction

The equations which govern sound propagation in flows are so complicated that analytic solutions exist only for the very simplest examples. Therefore there is much interest in global properties of the sound waves which are often of help in the comprehension of the solutions and offer secondly a valuable check in the assessment of numerical solutions. These properties are such that some of them, like conservation or symmetry principles, refer to a single solution and others like Babinet's principle or reciprocity with respect to interchange of source and observer relate two different solutions. Properties of one solution are, both algebraically and conceptually, simpler than properties of two solutions. The question of whether a given physical system conserves energy is a fundamental one and usually easy to decide. The same is true concerning time-reversal invariance. The situation concerning reciprocity is rather different. I have no clear physical understanding of the meaning of reciprocity. So it seems that the easiest explanation of the reciprocity property is obtained if one can relate it to energy conservation and time-reversal invariance. Relations between these properties have also been studied e.g. by Bojarsky (1983), Wapenaar (1994), Wapenaar & Grimbergen (1996) and Godin (1997) and in further papers quoted therein.

The problem of sound propagation in a non-uniform duct is important and seems very well suited to illustrate these relations. There are, for a given frequency, only a finite number of energy carrying modes and the scattering process can be described by a finite matrix, namely the scattering matrix. The three properties mentioned above can then be converted to matrix equations and the derivation of these relations is the main goal of this paper. It is obvious that these equations are such that the validity of two of them implies the validity of the third. Because of their importance in the applications, we also consider walls with losses and describe the modifications which occur. In the first part (§ 2) we assume a potential flow in the duct. We can then base our arguments on the Blokhintzev energy equation. In the seond part we assume a rotational flow in the duct. There, the energy equation contains potentials. Then modifications are required, such that only the reciprocity relation remains valid.

The configuration which we consider consists of two three-dimensional semi-infinite uniform ducts with arbitrary cross-sections which are connected by a smooth transition region. The walls may be rigid or compliant. We show that the relations which were derived before only for rigid walls or for compliant walls without mean flow can be extended to ducts with compliant walls and with mean flow. During the main part of this paper we assume that the walls are lossless, because of the much more powerful methods which are available then. Of course energy conservation and time-reversal invariance require lossless walls. Often, and that is the only situation which we will consider, one can include losses by assuming that material properties which are real for a lossless situation become complex. Then the principle of analytic continuation allows one to infer from the validity of an analytic equation for all real values its validity for complex values. The mathematical methods which we use are based on the scattering matrix formulation, because the interdependence of the three properties is easiest to visualize there. We will however also show that the vanishing of an antisymmetric bilinear expression, which is often called self-adjointness and which is usually used to show reciprocity in free space, see e.g. Pierce (1981), or in ducts without flow by Eversman (1976) and Cho (1980) is actually a disguised energy conservation statement, and therefore easy to obtain, once energy conservation and time-reversal invariance are available.

In the configuration with two semi-infinite ducts there are the infinite duct modes in both half-ducts and relations between the modal amplitudes in both half-ducts are obtained from the energy conservation and from the reciprocity principle. They have to be fulfilled exactly and they therefore provide a valuable check for numerical computations. In physical applications one often considers walls with losses. The reciprocity relation, which remains valid, is therefore often of more significance than the energy conservation theorem with its restricted validity. Because of their close connection it seems appropriate however to consider both principles simultaneously.

In the first part we consider irrotational sound waves in ducts with flow. This problem has recently been studied using the method of multiple scales by Rienstra (1999). There one has the Blokhintzev energy equation. This equation is a local one which is converted to a global one of modal energy conservation. This can be achieved for rigid walls or for compliant walls if the mean flow vanishes at the wall or if the wall impedance varies continuously. For discontinuous liner properties, the validity of modal energy conservation depends on the edge conditions. Various conditions in the range from finite liner displacements to a Kutta condition have been used in the literature, see e.g. Koch & Möhring (1983), Möhring & Eversman (1982), Howe (1998). Then the inter-relation between energy conservation, time-reversal invariance and reciprocity can be discussed. As the time-reversal invariance requires the reversal of the flow velocity, there is a reciprocity with respect to an interchange of source and observer if at the same time the flow velocity is reversed. The validity of such a theorem was apparently first proved for a uniform flow velocity by Lyamshev

(1961) and for potential flows to first order in Mach number by Howe (1975) and for arbitrary Mach number in Möhring (1978).

Then we consider rotational flow. Sound waves propagating in such a flow are also rotational. A velocity potential does not exist for them. This implies also that there is a coupling between acoustical and hydrodynamic modes. Therefore both have usually to be taken into acount. Several differential energy conservation theorems have been derived by Möhring (1970), Andrews & McIntyre (1978), and Godin (1996*b*). They are not formulated in the primitive fluid dynamic variables, i.e. particle velocity, pressure and density, but make use of additional quantities. Möhring (1970) uses Clebsch potentials while particle displacements in a mixed Eulerian–Lagrangian description occur in Andrews & McIntyre (1978) and Godin (1996*b*). The last two works consider the much more general problem of waves in an arbitrary flow of a compressible heavy fluid in an inhomogeneous gravity field. In primitive variables alone, only energy balance equations have been derived by e.g. Morfey (1967, 1971), Fuchs (1969) and Myers (1991). They involve energy sources and state a balance between the acoustic energy generated in some region by the sources and that leaving it through its surface. The applications which we have in mind however require energy conservation without sources. Apparently contradicting statements have been made in this case. While in Möhring (1978) it has been wtated that global energy conservation and reciprocity is not fullfilled in ducts with rotational mean flow, global energy conservation and reciprocity are claimed to be valid in arbitrary rotational flow by Godin (1996*a, b*). Therefore it seems useful to reconsider the situation. One can derive the result that the energy fluxes in both half-ducts cancel, under the same conditions as in the irrotational case, i.e. for rigid or compliant lossless walls provided the wall properties change continuously. Then total energy conservation and reciprocity seem to be fulfilled. However, the quantities contain the Clebsch potentials or the particle displacements respectively. The conflicting statements originate from a different assessment of this fact. While Godin (1996*a, b*) considers the particle displacements as observable quantities, Möhring (1978) notes that the Clebsch potentials – and also the particle displacements – in both half-ducts cannot be determined from the primitive variables in the respective half-ducts, but require for their determination knowledge of the flow-field in the whole duct. Therefore a relation between the primitive gasdynamic variables in both half-ducts is not obtained from the energy and reciprocity theorem. A reconsideration of the situation shows that this is actually true for the energy conservation theorem. It is however possible to select the Clebsch potentials in such a manner that a reciprocity relation formulated in terms of the primitive variables holds. This leads to the somewhat strange situation that time-reversal invariance and reciprocity are fulfilled but not energy conservation. It is closely related to the non-uniqueness of the Clebsch potentials.

2. Irrotational sound waves

We consider a duct carrying a steady potential flow of an ideal lossless gas of velocity $\boldsymbol{v}_0(\boldsymbol{x})$, density $\rho_0(\boldsymbol{x})$, pressure $p_0(\boldsymbol{x})$ and speed of sound $a_0(\boldsymbol{x})$. We assume that the duct consists of two uniform sections which are connected in $|x| < x_t$ by a smooth transition region (figure 1) and that all flow quantities are constant in the uniform sections. The walls may be rigid or compliant, but we assume that the wall properties are also constant in the uniform duct sections $|x| > x_t$. Additionally we consider a sound wave in the duct, which is described by an acoustic potential ϕ, where the

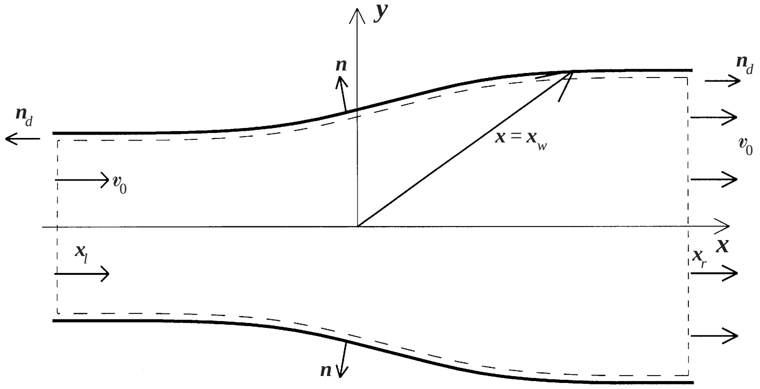

FIGURE 1. Schematic sketch of the non-uniform duct with the control surface.

particle velocity $\boldsymbol{v}$, and the pressure p, and density ρ are given by

$$\boldsymbol{v} = \nabla\phi, \quad p = -\rho_0\frac{\mathrm{D}\phi}{\mathrm{D}t}, \quad \rho = -\frac{\rho_0}{a_0^2}\frac{\mathrm{D}\phi}{\mathrm{D}t} \quad \text{with} \quad \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_0\cdot\nabla. \tag{1}$$

From the continuity equation, and with the definitions from (1) the governing equation for ϕ is found:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}_0 + \rho_0\boldsymbol{v}) = 0. \tag{2}$$

On multiplying (2) with ϕ_t the acoustic energy equation is obtained:

$$\frac{\partial w(\psi)}{\partial t} + \nabla\cdot\boldsymbol{U}(\psi) = 0 \quad \text{with} \quad w = \frac{\rho_0}{2}\boldsymbol{v}^2 + \frac{p^2}{2\rho_0 a_0{}^2} + \frac{\boldsymbol{v}_0\cdot\boldsymbol{v}p}{a_0{}^2}, \quad \boldsymbol{U} = -\phi_t(\rho_0\boldsymbol{v} + \rho\boldsymbol{v}_0). \tag{3}$$

Here ψ denotes a vector which contains all acoustic quantities $\boldsymbol{v}, p, \rho, \phi, \ldots$. It seems interesting that the quantities in (1) are also well defined if the mean flow is not irrotational and (2), (3) are valid in that case too, if the unperturbed quantities are independent of t. In this situation (3) has no obvious physical meaning, but it might be suitable as a test case for numerical schemes. Instead of the isentropic Euler equations, $\boldsymbol{v}$ and p then fulfil the equation

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}_0 + \nabla\frac{1}{\rho_0}p = \boldsymbol{v}\times(\nabla\times\boldsymbol{v}_0)$$

with an unphysical additional vorticity term on the right-hand side. Furthermore we consider harmonic time dependence such that all sound quantities are proportional to $\mathrm{e}^{-\mathrm{i}\omega t}$, where the physical quantities are the real parts, i.e. $\psi \to \frac{1}{2}(\psi + \psi^*)$, and where ψ^* denotes the conjugate complex of ψ. It is convenient to introduce in addition to $\boldsymbol{U}(\psi)$ the bilinear expression

$$\begin{aligned}\boldsymbol{U}(\psi_1, \psi_2) &= \tfrac{1}{4}(\boldsymbol{U}(\psi_1 + \psi_2) - \boldsymbol{U}(\psi_1 - \psi_2)) \\ &= -\tfrac{1}{2}(\phi_{1t}(\rho_0\boldsymbol{v}_2 + \rho_2\boldsymbol{v}_0) + \phi_{2t}(\rho_0\boldsymbol{v}_1 + \rho_1\boldsymbol{v}_0))\end{aligned} \tag{4}$$

and a similar one for w. We will always write the two arguments explicitly, if we consider the bilinear expression. $\boldsymbol{U}$ with one argument or without an argument refers to the energy flux $\boldsymbol{U}(\psi) = \boldsymbol{U}(\psi, \psi)$. The same is true for w or for components of $\boldsymbol{U}$. Then also

$$\frac{\partial w(\psi_1, \psi_2)}{\partial t} + \nabla\cdot\boldsymbol{U}(\psi_1, \psi_2) = 0$$

and for the mean energy $\overline{\boldsymbol{U}(\psi)} = \frac{1}{2}\boldsymbol{U}(\psi, \psi^*)$ with $\psi_1 = \psi$ and $\psi_2 = \psi^*$ the equation

$$\nabla \cdot \boldsymbol{U}(\psi, \psi^*) = 0$$

is obtained as $w(\psi, \psi^*)$ is independent of t for harmonic time dependence. Let us assume that the duct walls are compliant and that the position vectors of the undisplaced and displaced duct walls are described by

$$\boldsymbol{x} = \boldsymbol{x}_w(x, \sigma), \quad \boldsymbol{x} = \boldsymbol{x}_w(x, \sigma) + \eta(t, x, \sigma)\boldsymbol{n}$$

where σ is a parameter on the curve obtained by the cross-section of the duct with the plane $x = \text{const}$. We assume that the duct is uniform for $|x| > x_t$, i.e. $\boldsymbol{x}_w$ is independent of x there. The variable η denotes the normal displacement of the perturbed duct wall. Furthermore a linear relation between the wall pressure and the wall displacement

$$p(t, \boldsymbol{x}_w) = L(\boldsymbol{x}_w)\,\eta(t, \boldsymbol{x}_w) \tag{5}$$

with an impedance $L(\boldsymbol{x}_w)$, which may depend on the wall position $\boldsymbol{x}_w$, is assumed. L should be constant in the uniform duct region $|x| > x_t$. The boundary condition which expresses the fact that the wall consists for all times of the same fluid particles was derived by Myers (1980), namely

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} = \boldsymbol{n} \cdot (\nabla\phi + \eta\boldsymbol{n} \cdot \nabla\boldsymbol{v}_0)|_{\boldsymbol{x}=\boldsymbol{x}_w} \tag{6}$$

where it is assumed that the unperturbed velocity is in the direction of the unperturbed wall $\boldsymbol{n} \cdot \boldsymbol{v}_0 = 0$. If the walls are purely mass- or spring-like the impedances L are real. Losses can be included if one considers complex impedances. As the available mathematical methods are much more powerful for lossless systems, we will consider these. Some of the results obtained for lossless systems, but by no means all, can then by an analytic continuation argument be carried over from real values of the L to complex values. Although such an argument, may seem weak, it at least gives a hint of what to expect for walls with losses and how to derive the solution eventually by an independent argument.

We now integrate the energy equation (3) over the control surface of figure 1. (For two-dimensional ducts integration in the spanwise direction can be omitted.) The integrals over the duct cross-section at x_l and x_r are in the uniform duct section. We then obtain

$$\int_{x=x_r} \overline{U_1(\psi)}\,\mathrm{d}A - \int_{x=x_l} \overline{U_1(\psi)}\,\mathrm{d}A + \int_{\text{Wall}} \overline{\boldsymbol{U}(\psi)} \cdot \boldsymbol{n}\,\mathrm{d}A = 0 \tag{7}$$

where the overbar denotes the time average. With (3) the energy flux into the wall is

$$\begin{aligned} -\bar{\boldsymbol{U}} \cdot \boldsymbol{n} &= \rho_0\overline{\phi_t\boldsymbol{n} \cdot \nabla\phi} \\ &= \rho_0\overline{\phi_t\left(\frac{\mathrm{D}\eta}{\mathrm{D}t} - \eta\boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla)\boldsymbol{v}_0\right)} = \rho_0\frac{\mathrm{D}\overline{\phi_t\eta}}{\mathrm{D}t} - \rho_0\overline{\eta\frac{\mathrm{D}\phi_t}{\mathrm{D}t}} - \rho_0\overline{\phi_t\eta}\boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla)\boldsymbol{v}_0 \\ &= \rho_0\boldsymbol{v}_0 \cdot \nabla\overline{\phi_t\eta} + \overline{\eta p_t} - \rho_0\overline{\phi_t\eta}\boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla)\boldsymbol{v}_0 = \nabla \cdot \overline{\phi_t\eta}\rho_0\boldsymbol{v}_0 - \boldsymbol{n} \cdot (\boldsymbol{n} \cdot \nabla)\rho_0\overline{\phi_t\eta}\boldsymbol{v}_0, \end{aligned}$$

where the pressure has been introduced using (1) and use has been made of the fact that the mean of a time derivative vanishes. The same argument leads, with the impedance relation (5), to a vanishing of the pressure-containing term in the last but one of the above expressions. The last equality is obtained with $\nabla \cdot \rho_0\boldsymbol{v}_0 = 0$ and with the vanishing of the normal velocity $\boldsymbol{v}_0 \cdot \boldsymbol{n} = 0$ at the wall. This can be written as a double vector product, finally giving

$$\bar{\boldsymbol{U}}\cdot\boldsymbol{n} = -\boldsymbol{n}\cdot(\nabla\times(\boldsymbol{n}\times\overline{\phi_t\eta}\rho_0\boldsymbol{v}_0)). \tag{8}$$

Originally the ∇-operator did not act on the $\boldsymbol{n}$-factor in the vector product, but it is easy to check that there are no additional contributions if one applies it to that factor too.

If the acoustic quantities at the duct walls are continuous one can use Stokes' theorem to tranform the surface integral over the duct walls in (7) with the integrand taken from (8) into a line integral over its boundary. In the case considered here, the boundary consists of two curves at the cross-sections of the duct with the planes $x = x_l$ and $x = x_r$, and one obtains therefore two line integrals over these curves. If one introduces $\boldsymbol{t}$ as the (right-handed) tangential vector of these curves, one obtains

$$\begin{aligned}\int_{\text{Wall}} \bar{\boldsymbol{U}}\cdot\boldsymbol{n}\,\mathrm{d}A &= \oint_{x=x_r}(\boldsymbol{t}\times\boldsymbol{n})\cdot\overline{\phi_t\eta}\rho_0\boldsymbol{v}_0\,\mathrm{d}s + \oint_{x=x_l}(\boldsymbol{t}\times\boldsymbol{n})\cdot\overline{\phi_t\eta}\rho_0\boldsymbol{v}_0\,\mathrm{d}s \\ &= \oint_{x=x_r}\overline{\phi_t\eta}\rho_0\boldsymbol{v}_0\,\mathrm{d}s\cdot\boldsymbol{n}_d + \oint_{x=x_l}\overline{\phi_t\eta}\rho_0\boldsymbol{v}_0\,\mathrm{d}s\cdot\boldsymbol{n}_d.\end{aligned} \tag{9}$$

Because of the right-handedness, the vectors $\boldsymbol{t}$ at $x = x_l$ and $x = x_r$ are in opposite directions. Therefore $\boldsymbol{n}_d = \boldsymbol{t}\times\boldsymbol{n}$ denotes the outer normal in the respective uniform duct sections. Then from (7) and (9) a difference over two contributions from $x = x_l$ and $x = x_r$ is obtained:

$$\overline{U_{tot}(\psi)}|_{x=x_r} - \overline{U_{tot}(\psi)}|_{x=x_l} = 0 \quad \text{with} \quad \overline{U_{tot}(\psi)} = \int \overline{U_1(\psi)}\,\mathrm{d}A - \oint \overline{\phi_t\eta}\rho_0 u_0\,\mathrm{d}s. \tag{10}$$

Energy conservation states that the mean of the total energy flux

$$U_{tot}(\psi,\psi^*)|_{x=x_r} = U_{tot}(\psi,\psi^*)|_{x=x_l} \tag{11}$$

is independent of the position considered. Here as in (4), we have used the corresponding bilinear expression

$$U_{tot}(\psi_1,\psi_2) = \int U_1(\psi_1,\psi_2)\,\mathrm{d}A - \tfrac{1}{2}\oint(\phi_{1t}\eta_2 + \phi_{2t}\eta_1)\rho_0 u_0\,\mathrm{d}s. \tag{12}$$

If the acoustic quantities are only piecewise continous Stokes' theorem can be applied to the continuous sections. The most important case is a finite length liner from $x = x_0$ to $x = x_1$ in an otherwise rigid walled duct. Then the surface integral over the duct walls does not depend on x_l and x_r if x_0 and x_1 are between x_l and x_r. Then instead of (11)

$$\overline{U_{tot}(\psi)}|_{x=xr} - \overline{U_{tot}(\psi)}|_{x=xl} = \oint_{x=x_1}\overline{\phi_t\eta}\rho_0\boldsymbol{v}_0\,\mathrm{d}s\cdot\boldsymbol{n}_d + \oint_{x=x_0}\overline{\phi_t\eta}\rho_0\boldsymbol{v}_0\,\mathrm{d}s\cdot\boldsymbol{n}_d. \tag{13}$$

There is no energy conservation, if the line integrals of (13) differ from zero. It is obvious that they vanish if the mean flow velocity $\boldsymbol{v}_0$ vanishes at x_0 and x_1. They would also vanish if the liner displacements vanish at the edges of the liner. It is not very clear which edge conditions are physically appropriate in this situation. In Koch & Möhring (1983) it has been shown that ambiguities originating from various possible edge conditions arise also in the special case of a finite length uniform liner in a uniform duct with uniform flow. There it was possible to require vanishing displacement at the edge. This would again lead to energy conservation. This is in contrast to other edge conditions studied in Koch & Möhring (1983) which would not show energy conservation. Further research is required to identify physically realistic edge conditions.

Let us now return to a situation with modal energy conservation (11). We evaluate U_{tot} in the uniform duct section. There we can write all sound field quantities as a superposition of the uniform duct modes

$$\psi_\nu = \Psi_\nu(y)\,\mathrm{e}^{\mathrm{i}(k_\nu x-\omega t)},$$

some from the upstream and some from the downstream uniform sections. Then

$$\psi = \sum_\nu A_\nu \Psi_\nu(y)\,\mathrm{e}^{\mathrm{i}(k_\nu x-\omega t)} \tag{14}$$

and from (11)

$$U_{tot}(\psi,\psi^*) = \sum_{\nu,\mu} A_\nu A_\mu^* U_{tot}(\Psi_\nu,\Psi_\mu^*)\mathrm{e}^{\mathrm{i}(k_\nu-k_\mu)x_{l,r}} \tag{15}$$

where $x_{l,r}$ denotes the x-coordinate of the respective integration surface, i.e. x_l for modes propagating in the left half and x_r for modes in the right half, and the summation is only over modes propagating in the considered half-duct. This has to be independent of x_l and x_r. So the total interaction energy flux $U_{tot}(\Psi_\nu,\Psi_\mu^*)$ has to vanish if $k_\nu \neq k_\mu$. If there are several modes having the same k_ν one can always choose them to be orthogonal to each other. So one has to have

$$U_{tot}(\Psi_\nu,\Psi_\mu^*) = u_\nu\,\delta_{\nu\mu} \tag{16}$$

with some real constants u_ν. Here $\delta_{\nu\mu}$ denotes the Kronecker symbol which vanishes for $\nu \neq \mu$ and is equal to 1 otherwise. This is a kind of orthogonality for the modes. We have derived it for real L. We now consider the cut-on modes having real k_ν. It is easy to check that the differential equation for the mode shape function of the potential $\Phi_\nu(y)$, which is obtained from (2), is real for real k_ν and real ω. The boundary conditions which are obtained from (5), (6) are also real for lossless walls, i.e. for real L. Therefore the mode shape functions of the potential $\Phi_\nu(y)$ are also real for lossless walls. One can then write $U_{tot}(\Psi_\nu,\Psi_\mu^*) = U_{tot}(\Phi_\nu,\Phi_\mu)$ and therefore from (15)

$$U_{tot}(\psi,\psi^*) = \sum_\nu A_\nu A_\nu^* U_{tot}(\Phi_\nu,\Phi_\nu) \tag{17}$$

as the Φ_ν are real. Here one can apply the analytic continuation argument. Equations (17) and (12) show that U_{tot} is an analytic function of Φ_ν and therefore also of the L. Let us illustrate this in more detail for a frequency such that there are one upstream and one downstream propagating mode in both uniform half-ducts. An incident mode Φ_1 from the left would then generate a reflected mode $R\,\Phi_2$ in the left half-duct and a transmitted mode $T\,\Phi_3$ in the right half-duct. Then from (17)

$$U_{tot}(\Phi_1,\Phi_1) + RR^* U_{tot}(\Phi_2,\Phi_2) = TT^* U_{tot}(\Phi_3,\Phi_3). \tag{18}$$

This also explains Rienstra's (1999) complete integration of the governing equations for sound propagation in slowly varying ducts. Rienstra's multiple scales argument shows that there is no reflected mode generated, $R = 0$. Therefore there is only one propagating mode. For lossless walls the total modal energy flux has to be constant and this determines the modal amplitude. This agrees with Rienstra's principal equation (4.10), obtained by him as solvability condition for the second-order approximation.† An analytic continuation to walls with loss is possible if one uses Rienstra's result, that the transmission coefficient T is real.

We now return to lossless walls. Then the modes with positive u_ν have a positive outgoing energy flux. We denote them as $\psi_\nu^{(o)}$ with wavenumber $k_\nu^{(o)}$ and energy flux $u_\nu^{(o)}$ and the incoming modes with negative energy flux are denoted similarly by $\psi_\nu^{(i)}$, $k_\nu^{(i)}$, and $-u_\nu^{(i)}$. Both energy flux constants $u_\nu^{(o,i)}$ are then positive. The cut-off modes with complex k_ν have a vanishing energy flux. We consider the sound field generated by one incident acoustic mode $\psi_\nu^{(i)}$. The sound field in the uniform duct section will then additionally consist of outgoing modes $\psi_\mu^{(o)}$ of an amplitude $s_{\nu\mu}$, i.e.

$$\begin{aligned}\psi &= \psi_\nu^{(i)} + \sum_\mu s_{\nu\mu}\psi_\mu^{(o)} \\ &= \Psi_\nu^{(i)}(y)\exp(\mathrm{i}(k_\nu^{(i)}x - \omega t)) + \sum_\mu s_{\nu\mu}\Psi_\mu^{(o)}(y)\exp(\mathrm{i}(k_\mu^{(o)}x - \omega t)). \qquad (19)\end{aligned}$$

The matrix $s_{\nu\mu}$ is called the scattering matrix. Its determination is usually the main goal of numerical computations. Inserting this equation into the energy equation (15) for an incident mode ν and an incident mode μ, with the normalization constants $u_\nu^{(i)}$ and $u_\nu^{(o)}$ from (16) results in

$$-u_\nu^{(i)}\delta_{\nu\mu} + \sum_\lambda s_{\nu\lambda}u_\lambda^{(o)}s_{\mu\lambda}^* = 0. \qquad (20)$$

The explicit occurrence of contributions from both half-ducts in equation (11) implies that the summation in equation (20) extends over all cut-on modes. The scattering matrix formed with the cut-on modes only is finite. This matrix is unitary if the modes are normalized such that the energy flux constants $u_\nu^{(i)}$ and $u_\nu^{(o)}$ are unity. (In matrix notation $\boldsymbol{SS}^{*T} = \boldsymbol{I}$ with $\boldsymbol{S} = (s_{\nu\mu})$. Here $\boldsymbol{S}^*$ denotes the conjugate complex of $\boldsymbol{S}$, $\boldsymbol{S}^T$ its transpose, and $\boldsymbol{I}$ the identity matrix.)

Let us now consider the time-reversal properties. Equation (2) shows that a substitution $t \to -t$ leads to a solution of the same equation with the mean flow velocity reversed, $\boldsymbol{v}_0 \to -\boldsymbol{v}_0$. Reversing the time leads to a solution of the original equation of the form of (14) if one takes additionally the conjugate complex. As time reversal also reverses the energy flux, we find that every incoming mode is converted to an outgoing mode. We assume that the mode numbers have been chosen such that the νth incoming mode is converted to the νth outgoing one. We then obtain from the solution (19) of the original duct

$$\psi = \Psi_\mu^{(o)}(-\boldsymbol{v}_0)\exp(\mathrm{i}(k_\mu^{(o)}(-\boldsymbol{v}_0)x - \omega t)) + \sum_\lambda s_{\mu\lambda}^*\Psi_\lambda^{(i)}(-\boldsymbol{v}_0)\exp(\mathrm{i}(k_\lambda^{(i)}(-\boldsymbol{v}_0)x - \omega t)) \qquad (21)$$

a solution of the duct with reversed flow, provided

$$\Psi_\mu^{(i,o)}(-\boldsymbol{v}_0) = \Psi_\mu^{(o,i)^*}, \quad k_\mu^{(i,o)}(-\boldsymbol{v}_0) = -k_\mu^{(o,i)^*}. \qquad (22)$$

This solution contains only one outgoing mode. We obtain the solution which contains all outgoing modes with the correct amplitudes if we multiply equation (21) with the scattering matrix elements $s_{\nu\mu}(-\boldsymbol{v}_0)$ of the reversed flow and sum over all cut-on

† The comparison shows, that Rienstra's coefficient σ^2 is a complete square, i.e. with Rienstra's notation $\sigma^2 = (C_0\mu/\omega + (U_0/C_0)(1 - U_0\mu/\omega))^2$, a square root is therefore not needed in the determination of σ.

modes. Therefore

$$\tilde{\psi} = \sum_{\mu} s_{\nu\mu}(-\boldsymbol{v}_0)\Psi_{\mu}^{(o)}(y)\exp\left(\mathrm{i}(k_{\mu}^{(o)}x - \omega t)\right) + \sum_{\lambda,\mu} s_{\nu\mu}(-\boldsymbol{v}_0)s_{\mu\lambda}^{*}\Psi_{\lambda}^{(i)}(y)\exp\left(\mathrm{i}(k_{\lambda}^{(i)}x - \omega t)\right)$$

has the same outgoing cut-on modes as the sound field generated from the incident duct mode ν. Therefore

$$\sum_{\mu} s_{\nu\mu}(-\boldsymbol{v}_0)s_{\mu\lambda}^{*} = \delta_{\nu\lambda}. \tag{23}$$

($\boldsymbol{S}(-\boldsymbol{v}_0)\boldsymbol{S}^{*} = \boldsymbol{I}$ for normalized modes.) This equation is a reformulation of the time-reversal invariance in terms of the scattering matrix. If one now multiplies (20) with $s_{\kappa\mu}(-\boldsymbol{v}_0)$ and sums over all κ belonging to cut-on modes, one obtains with (23)

$$-u_{\nu}^{(i)}s_{\kappa\nu}(-\boldsymbol{v}_0) + s_{\nu\kappa}u_{\kappa}^{(o)} = 0. \tag{24}$$

($\boldsymbol{S}(-\boldsymbol{v}_0) = \boldsymbol{S}^{T}$ for normalized modes.) This is the reciprocity relation. If the modes are normalized to unit mean energy flux, the transpose of the scattering matrix is equal to the scattering matrix of the reversed flow. It is obvious that two of the three relations (20), (23), (24) imply the third. Notice however, that (20), (23) are not analytic functions of their elements; (24) is analytic. Therefore the validity of (24) for real wall impedances also implies the validity of (24) for complex wall impedances. Reciprocity is not restricted to lossless walls.

Usually reciprocity is derived from the symmetry or self-adjointness of the wave operator. A bilinear expression with vanishing divergence is derived. This bilinear expression is very similar to the interaction energy flux; however, it is not symmetric like the interaction energy flux but antisymmetric. Here we show that the reciprocity relation can also be derived directly from the symmetric interaction energy equation using the time invariance. We observe that the time-reversed solution $\psi^{(-)} = \psi(-\boldsymbol{v}_0)|_{t\to -t}$ from (19) has a time dependence proportional to $\mathrm{e}^{\mathrm{i}\omega t}$. This solution and the solution from (19) then have an interaction energy flux with vanishing divergence, i.e.

$$\nabla \cdot \boldsymbol{U}(\psi, \psi^{(-)}) = 0 \quad \text{and} \quad U_{tot}(\psi, \psi^{(-)})|_{x=x_l} = U_{tot}(\psi, \psi^{(-)})|_{x=x_r}. \tag{25}$$

Inserting the solution from (19) and from the same solution to the reversed flow leads with (22) and the mode orthogonality (16) immediately to

$$-u_{\nu}^{(i)}s_{\mu\nu}(-\boldsymbol{v}_0) + s_{\nu\mu}u_{\mu}^{(o)} = 0,$$

i.e. the reciprocity relation (24). So it seems that the usual antisymmetric bilinear relation is just a reformulation of the symmetric interaction energy equation using time-reversal invariance.

3. Rotational sound waves

3.1. *Clebsch potentials*

A sheared flow in a duct is not irrotational. Sound waves propagating in such a flow therefore also carry vorticity. A velocity potential does not exist for them. The energy conservation relations which have been derived use additional auxiliary variables, namely Clebsch potentials or particle displacements. Let us recapitulate briefly the Clebsch potential formulation from Möhring (1973). Introducing ϕ_0, η_0, α_0, and β_0 for the mean flow one has with the mean enthalpy h_0 and entropy s_0 the differential relation

$$\boldsymbol{v}_0 \cdot \mathrm{d}\boldsymbol{x} - (h_0 + \tfrac{1}{2}v_0^2)\,\mathrm{d}t = \mathrm{d}\phi_0 + s_0\,\mathrm{d}\eta_0 + \alpha_0\,\mathrm{d}\beta_0 \tag{26}$$

which means of course

$$\boldsymbol{v}_0 = \nabla\phi_0 + s_0\nabla\eta_0 + \alpha_0\nabla\beta_0, \quad (h_0 + \tfrac{1}{2}v_0^2) = -\frac{\partial\phi_0}{\partial t} - s_0\frac{\partial\eta_0}{\partial t} - \alpha_0\frac{\partial\beta_0}{\partial t}.$$

The potentials obey the equations

$$\frac{\mathrm{D}\eta_0}{\mathrm{D}t} = -T_0, \quad \frac{\mathrm{D}\alpha_0}{\mathrm{D}t} = 0, \quad \frac{\mathrm{D}\beta_0}{\mathrm{D}t} = 0.$$

For steady flows one may assume that only β_0 depends on the time t, namely $\beta_0 = B(x) - t$. Similarly, Clebsch potentials for the linearized flow with

$$\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} - (h + \boldsymbol{v}_0\cdot\boldsymbol{v})\,\mathrm{d}t = \mathrm{d}\phi + s_0\,\mathrm{d}\eta + s\,\mathrm{d}\eta_0 + \alpha_0\,\mathrm{d}\beta + \alpha\,\mathrm{d}\beta_0 \tag{27}$$

and

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} + \boldsymbol{v}\cdot\nabla\eta_0 = -T, \quad \frac{\mathrm{D}\alpha}{\mathrm{D}t} + \boldsymbol{v}\cdot\nabla\alpha_0 = 0, \quad \frac{\mathrm{D}\beta}{\mathrm{D}t} + \boldsymbol{v}\cdot\nabla\beta_0 = 0 \tag{28}$$

are introduced. Then one can derive an energy equation (3) with an energy flux

$$\boldsymbol{U} = h_{tot}(\rho_0\boldsymbol{v} + \rho\boldsymbol{v}_0) - \rho_0(s\eta_t + \alpha\beta_t)\boldsymbol{v}_0, \quad h_{tot} = h + \boldsymbol{v}_0\cdot\boldsymbol{v} - \alpha. \tag{29}$$

The energy flux depends on the Clebsch potentials. The Clebsch potentials are not uniquely determined, in general there are various energy fluxes and one has some freedom to choose the most convenient one. In Möhring (1973) it is shown that the mean energy flux is independent of the choice of the Clebsch potentials of the mean flow. So it remains to study the influence of the Clebsch potentials of the linearized flow. From (27) it can be concluded that canonical transformations may be used to obtain from one set of Clebsch potentials another one. Writing the differential form (27) with an arbitrary function $W = W(s_0, \alpha_0, \beta_0)$:

$$\begin{aligned} &\mathrm{d}(\phi + s_0\eta + \alpha_0\beta) - \eta\,\mathrm{d}s_0 + s\,\mathrm{d}\eta_0 - \beta\,\mathrm{d}\alpha_0 + \alpha\,\mathrm{d}\beta_0 \\ &= \mathrm{d}(\phi + s_0\eta + \alpha_0\beta + W) - \left(\eta + \frac{\partial W}{\partial s_0}\right)\mathrm{d}s_0 + s\,\mathrm{d}\eta_0 \\ &\quad - \left(\beta + \frac{\partial W}{\partial\alpha_0}\right)\mathrm{d}\alpha_0 + \left(\alpha - \frac{\partial W}{\partial\beta_0}\right)\mathrm{d}\beta_0, \end{aligned}$$

it can be seen that new Clebsch potentials $\tilde{\phi}$, $\tilde{\eta}$, $\tilde{\alpha}$, $\tilde{\beta}$ can be used with

$$\tilde{\eta} = \eta + \frac{\partial W}{\partial s_0}, \quad \tilde{\beta} = \beta + \frac{\partial W}{\partial\alpha_0}, \quad \tilde{\alpha} = \alpha - \frac{\partial W}{\partial\beta_0}. \tag{30}$$

The difference between the two sets of Clebsch potentials describes a ‘pseudo-flow’ with no effect on the primitive gasdynamic variables pressure, density and velocity. This pseudo-flow leads to a modified energy flux $\tilde{\boldsymbol{U}}$ however. In the special case, that W depends only on β_0, one obtains e.g.

$$\tilde{\boldsymbol{U}} = \boldsymbol{U} + \frac{\partial W}{\partial\beta_0}(\rho_0\boldsymbol{v} + \rho\boldsymbol{v}_0 + \beta_t\rho_0\boldsymbol{v}_0)$$

and more complicated expressions in the general case. They lead in general to different values of the time-averaged energy flux, depending on the choice of W. For each choice there is conservation of the mean energy flux

$$\oint \bar{\boldsymbol{U}}\cdot\mathrm{d}\boldsymbol{x} = 0.$$

Making use of the relation

$$\frac{\mathrm{D}h_{tot}}{\mathrm{D}t} = \frac{1}{\rho_0}\frac{\partial p}{\partial t}$$

from Möhring (1978), gives, very similarly to (8) for the normal component of the energy flux into the walls,

$$\bar{\boldsymbol{U}}\cdot\boldsymbol{n} = \boldsymbol{n}\cdot(\nabla\times(\boldsymbol{n}\times\overline{h_{tot}\eta\rho_0\boldsymbol{v}_0}))$$

and again the integrals over soft duct walls can be converted into line integrals. For continuous wall properties, one obtains very similarly to (10)

$$\overline{U_{tot}(\psi)}|_{x=x_r} = \overline{U_{tot}(\psi)}|_{x=x_l} \quad \text{with} \quad U_{tot}(\psi) = \int \overline{U_1(\psi)}\,\mathrm{d}A + \oint \overline{h_{tot}\eta_k}\rho_0 u_0\,\mathrm{d}s \tag{31}$$

and an equation very similar to (13) for a finite length liner in an otherwise rigid walled duct. Equation (29) shows that the energy flux contains the Clebsch potentials and the contributions in the sum of (31) depend on them. So one needs the Clebsch potentials in the uniform duct sections. They depend on the whole flow field. If they are known, for lossless walls an energy conservation relation and a reciprocity relation are obtained, exactly as for irrotational sound waves. The Clebsch potentials in the uniform duct sections enter into these relations.

3.2. *Uniform ducts*

Let us now describe the linearized flow and its Clebsch potentials for uniform ducts in more detail. This seems appropriate, as the presentation in the papers mentioned previously is mainly restricted to sound waves in two-dimensional ducts and the relation to the hydrodynamic modes has not been studied there. Because of the coupling between acoustic and hydrodynamic modes in the duct's transition region, the hydrodynamic modes have to be taken into account. So one may write in the uniform duct

$$p = p_{ac}(x,\boldsymbol{y}) + p_{hy}(x,\boldsymbol{y})$$

and similarly for all other primitive gasdynamic variables ρ, s, $\boldsymbol{v}$. Here quantities with a subscript *ac* or *hy* refer to the acoustic or hydrodynamic modes respectively x denotes the coordinate in the duct's axial direction, and $\boldsymbol{y}$ a two-dimensional vector in the cross-sectional plane. Again, harmonic time dependance is assumed throughout. We exclude unstable mean flow profiles such that p_{ac} and p_{hy} remain bounded. Furthermore we assume that the acoustic part can be written as a superposition of discrete modes

$$p_{ac} = \sum c_\nu P_\nu(\boldsymbol{y})\exp(\mathrm{i}(k_\nu x - \omega t)). \tag{32}$$

Only a finite number of the k_ν are real, the remaining complex ones describing exponentially decaying ('cut-off') modes. The hydrodynamic modes are such that the wave numbers contained in p_{hy} are different from those in p_{ac}, i.e.

$$\langle \mathrm{e}^{-\mathrm{i}k_\nu x} p_{hy}(x,\boldsymbol{y})\rangle = 0, \quad \text{for real acoustic } k_\nu, \tag{33}$$

where the brackets denote a spatial average over the axial coordinate. We do not however assume that p_{hy} possesses a modal expansion similar to (32). In general a continuous spectrum has to be admitted in the hydrodynamic modes. This can be shown, at least for rectangular and circular ducts, by Fourier transform methods using the ideas described in Friedman (1956), pp. 217–250. Finally we require that the acoustic wavenumbers k_ν do not give rise to a critical layer, i.e. $\omega - k_\nu u_0(\boldsymbol{y}) \neq 0$ for all $\boldsymbol{y}$ in the duct crosssection, where $u_0(\boldsymbol{y})$ denotes the mean flow profile; the critical layers are associated with the hydrodynamic modes.

A set of Clebsch potentials can be obtained from (26), (28). For the mean flow one easily obtains

$$\eta_0 = -\frac{T_0(\boldsymbol{y})}{u_0(\boldsymbol{y})}x, \quad \alpha_0 = h_0(\boldsymbol{y}) + \tfrac{1}{2}u_0(\boldsymbol{y})^2, \quad \beta_0 = \tfrac{x}{u_0(\boldsymbol{y})} - t.$$

The Clebsch potentials of the linearized flow can be obtained from (28). For the acoustic part one may assume an expansion into a series of discrete modes, i.e. for α

$$\alpha_{ac} = \sum c_\nu A_\nu(\boldsymbol{y})\mathrm{e}^{\mathrm{i}(k_\nu x - \omega t)}$$

and for the A_ν

$$A_\nu = \frac{1}{\mathrm{i}(\omega - k_\nu u_0(\boldsymbol{y}))}\boldsymbol{v}_{ac}\cdot\frac{\partial\alpha_0}{\partial\boldsymbol{y}} = -\frac{1}{\rho_0(\omega - k_\nu u_0(\boldsymbol{y}))^2}\frac{\partial\alpha_0}{\partial\boldsymbol{y}}\cdot\frac{\partial P_\nu}{\partial\boldsymbol{y}}.$$

Also from (28) for the hydrodynamic modes

$$-\mathrm{i}\omega\alpha_{hy} + u_0\frac{\partial\alpha_{hy}}{\partial x} = -\boldsymbol{v}_{hy}\cdot\frac{\partial\alpha_0}{\partial\boldsymbol{y}}. \tag{34}$$

On multiplying this equation with $\mathrm{e}^{-\mathrm{i}k_\nu x}$, averaging over the axial coordinate and taking (33) into account, one obtains

$$\langle \mathrm{e}^{-\mathrm{i}k_\nu x}\alpha_{hy}\rangle = 0, \tag{35}$$

i.e. the very plausible result that α_{hy} does not contain acoustic wavenumbers. The situation is slightly more complicated for η and β as η_0 and β_0 are linearly growing with x. However, assuming

$$\eta = x\tilde{\tilde{\eta}} + \tilde{\eta}$$

gives the two equations

$$\frac{\mathrm{D}\tilde{\tilde{\eta}}}{\mathrm{D}t} = \boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{y}}\frac{T_0(\boldsymbol{y})}{u_0(\boldsymbol{y})}, \qquad \frac{\mathrm{D}\tilde{\eta}}{\mathrm{D}t} + u_0\tilde{\tilde{\eta}} - u\frac{T_0(\boldsymbol{y})}{u_0(\boldsymbol{y})} = -T. \tag{36}$$

Then the acoustic and hydrodynamic parts can be determined as for α and again the acoustic parts of $\tilde{\tilde{\eta}}$ and $\tilde{\eta}$ can be found as a superposition of discrete modes and their hydrodynamic parts do not contain acoustic wavenumbers.

Note that the mode series asumption for the acoustic modes determines the Clebsch potentials uniquely from the primitive variables and one can verify that this set of Clebsch potentials fulfils the defining relations (27). The hydrodynamic solutions are however not uniquely determined from the partial differential equations (34), (36) and a similar one for β. Furthermore solutions of these equations do not automatically satisfy (27); they have to be chosen appropriately. This choice is not unique. There are many possibilities related by the canonical transformations (30). This means e.g. that there are non-vanishing Clebsch potentials of the type of a hydrodynamic mode for a vanishing flow, which we called a pseudo-flow.

Now we can determine the energy flux in the duct from (31). We again introduce an energy flux $\boldsymbol{U}(\psi_1, \psi_2)$ similar to (4) and obtain from the energy equation that the temporal average of the total energy flux $U_{tot}(\psi, \psi^*)$ is independent of the axial position of the cross-section considered. For a flow with acoustic and hydrodynamic modes $\psi = \psi_{ac} + \psi_{hy}$, one obtains

$$U_{tot}(\psi, \psi^*) = U_{tot}(\psi_{ac}, \psi_{ac}^*) + U_{tot}(\psi_{hy}, \psi_{ac}^*) + U_{tot}(\psi_{ac}, \psi_{hy}^*) + U_{tot}(\psi_{hy}, \psi_{hy}^*).$$

As U_{tot} is independent of the axial position, the linearly growing terms $\tilde{\tilde{\eta}}$ and $\tilde{\tilde{\eta}}$ cannot enter U_{tot}; they have to cancel. For the same reason one can average over all axial positions. Then because of (33) and (35) the interaction energy flux between the acoustic and hydrodynamic modes vanishes. Similarly there is no interaction energy flux between different acoustic modes. We can therefore write for the acoustic field of (32)

$$U_{tot}(\psi, \psi^*) = \sum c_\nu c_\nu^* u_\nu + U_{tot}(\psi_{hy}, \psi_{hy}^*), \quad \text{with} \quad U_{tot}(\Psi_\nu, \Psi_\mu{}^*) = u_\nu\, \delta_{\nu\mu}, \tag{37}$$

where the modal energy flux can be expressed in terms of the pressure mode shape function

$$U_{tot}(\psi_\nu, \psi_\nu^*) = \frac{\omega}{2\rho_0 \Omega_\nu^2} \left[-\frac{\omega}{\Omega_\nu^2} \frac{\partial u_0}{\partial \boldsymbol{y}} \cdot P_\nu \frac{\partial P_\nu}{\partial \boldsymbol{y}} + \left(\frac{\Omega_\nu u_0}{a_0^2} + k_\nu \right) P_\nu^2 \right]. \tag{38}$$

Here $\Omega_\nu = \omega - k_\nu u_0(\boldsymbol{y})$. Equation (37) shows that the energy flux depends on the hydrodynamic modes. As the Clebsch potentials of these modes are not uniquely determined, the same is true of the energy flux. This is even true in a flow without hydrodynamic modes, there are non-vanishing Clebsch potentials which we called pseudo-flow. We are, however, free in an infinite duct without hydrodynamic modes to assume vanishing Clebsch potentials. Then the energy flux is well defined, and it is this definition of the energy flux which has been used before in purely acoustic flow in uniform ducts.

3.3. *Transitional ducts*

Let us now consider an inhomogeneous duct consisting of two uniform semi-infinite ducts, which are connected by some transition region similar to the configuration of figure 1 considered in § 2. We assume now, however, that the mean flow is not irrotational. Instead of Blokhintsev's energy equation (3), which is not applicable for rotational flows, we want to make use of the energy flux (31). We are again interested in relations between the sound fields in the two semi-infinite ducts. The energy flux (31) contains the Clebsch potentials and relations derived from it will generally also contain these potentials. There are two difficulties connected with this approach. First, the Clebsch potentials are not uniquely determined and secondly the Clebsch potentials do not belong to the primitive gasdynamic variables. The first difficulty is a minor one, as one gets relations for every choice of the potentials or one could eventually determine them uniquely by some additional condition. On the other hand, we consider the second difficulty a serious one. The sound field is completely determined by the primitive variables, and one can determine the Clebsch potentials from them. This is not locally possible however; one has to solve the differential equations (28). This means that one has to know the primitive variable everywhere in the duct, i.e. also in the transition region, to determine Clebsch potentials in the two uniform half-ducts. Assuming this, one can probably derive relations similar to those of § 2 for potential flows with complications associated with the existence of hydrodynamic modes and pseudo-flows. The usefulness of such an approach appears doubtful. We therefore restrict ourselves to relations which do not involve Clebsch potentials. It may be surprising that such relations exist, but we will show that the reciprocity relation (24) is also valid in rotational mean flow, the modal energy flux constants $u_\nu^{(i,o)}$ now determined from the well-defined energy flux (37).

To obtain the reciprocity relation, we assume that there are no incoming hydrodynamic modes: the linear flow is generated by an incoming acoustic mode. Then there are no hydrodynamic modes in the upstream uniform half-duct: only acoustic modes are found there. Then we can determine the Clebsch potentials there as a superposition of discrete modes. In the downstream half-duct, there will in general also be hydrodynamic modes which were generated from the incoming sound wave in the transition region. The same is true for the time-reversed solution $\psi^{(-)} = \psi(-\boldsymbol{v}_0)|_{t\to -t}$. Then we confirm again the validity of (25)

$$U_{tot}(\psi, \psi^{(-)})|_{x=x_l} = U_{tot}(\psi, \psi^{(-)})|_{x=x_r}. \tag{39}$$

Now it is important to recognize that the upstream half-duct of the original flow is the downstream half-duct of the reversed flow. Only one argument in (39), namely $\psi^{(-)}$ in the originally upstream half-duct and ψ in the originally downstream half-duct, contains hydrodynamic modes. As we have shown in § 3.2 that the total interaction energy flux between acoustic and hydrodynamic modes vanishes, only acoustic contributions remain in (39) and (24) is valid again.

Here one has obviously a situation where the relation between the three properties energy conservation, time-reversal invariance and reciprocity described above does not hold. Time-reversal invariance and reciprocity are fulfilled but not energy conservation. Let us conclude with the remark that the pressure mode shape function is real for real L and one can write, similar to (17)

$$U_{tot}(P_\nu, P_\mu) = u_\nu\, \delta_{\nu\mu}.$$

The remark after (24) is relevant and an analytic continuation to complex L is again possible. Reciprocity is also valid for walls with losses.

4. Conclusion

It has been shown that there is a close relation between energy conservation, time-reversal invariance and reciprocity with respect to interchange of source and observer for sound waves propagating in an inhomogeneous duct with rigid or soft walls which carries a potential flow. The scattering matrix is then finite and the three properties are translated into matrix relations. It is obvious that these relations are such that the validity of two of them implies the validity of the third. All three properties are fulfilled for lossless walls with continuous wall properties and for discontinuous liner properties if certain edge conditions are fulfilled. It is not very clear which edge conditions are physically significant. An analytic continuation argument shows that the reciprocity relation remains true for walls with loss with continuously varying properties.

For rotational flows there are energy conservation and reciprocity relations which depend on auxiliary variables, like Clebsch potentials or particle displacements. In addition a reciprocity relation which depends only on the primitive gasdynamic variables is also derived.

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